# MOTION OF A BODY IN THE CASE OF L.N. SRETENSKII

PMM Vol. 32, No. 3, 1968, pp. 535-542

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(Received January 29, 1968)

One of the most general cases of integrability of the Euler equations is that of Hess [1]. Sretenskii [2 and 3] generalized this result for the case of a gyrostat, and, taking the Euler angles as the variables, found that the angles of precession and nutation are elliptic functions of time. He also obtained a linear second-order differential equation with biperiodic coefficients for determining the angle of characteristic rotation.

We investigate this problem in the special coordinate system proposed by Kharlamov [4 and 5]. This simplifies our investigation considerably and enables us to reduce the problem to a simpler linear second-order differential equation. A case where this equation is integrable in elementary functions is indicated; the geometric picture of motion of a body in this case is described.

**1.** As was shown by Kharlamov [6], the conditions of existence of the Sretenskii solution and the fourth integral of the problem expressed in the special coordinate system are

$$a_2 = a_1 = a_*, \quad \lambda_1 = (b_1 / a_*)n, \quad \lambda_2 = 0, \quad b_2 = 0; \quad x = n$$

The equations and integrals of motion are

$$dy / dt = (an + b_1 y)z - a_* \, z\,(n + \lambda) - \nu_2 \Gamma$$

$$dz / dt = -(an + b_1 y)(y + \lambda_1) + (a_* y + b_1 n)(n + \lambda) + \nu_1 \Gamma$$

$$\tfrac{1}{2}[n(an + b_1 y) + y(bn + a_* y) + a_* z^2] - \nu\Gamma = E$$

$$(n + \lambda)\nu + (y + \lambda_1)\nu_1 + z\nu_2 = k, \qquad (1.1)$$

$$\nu^2 + \nu_1^2 + \nu_2^2 = 1$$

For $b_1 = 0$ we have the Lagrange case, so that we assume from now on that $b_1 \neq 0$. Let us introduce the dimensionless variables $y'$, $z'$, $\tau$, setting

$$y + \lambda_1 = \sqrt{\Gamma / b_1}\, y', \qquad z = \sqrt{\Gamma / b_1}\, z', \qquad t = \tau / \sqrt{\Gamma b_1}$$

We also set

$$d' = \frac{an - b_1\lambda_1 - a_*(n + \lambda)}{\sqrt{\Gamma b_1}}, \qquad c' = 2\frac{b_1}{a_*}, \qquad k' = k\left(\frac{b_1}{\Gamma}\right)^{1/2} \qquad (1.2)$$

$$h' = \frac{2E - n(an - b_1\lambda_1)}{2\Gamma}, \qquad e' = \left(\frac{b_1}{\Gamma}\right)^{1/2}(n + \lambda)$$

Eqs. (1.1) can now be rewritten as

$$y^{\cdot} = z(d + y) - \nu_2, \quad z^{\cdot} = -y(d + y) + \nu_1 \qquad (1.3)$$

$$y^2 + z^2 - c\nu = ch, \quad e\nu + y\nu_1 + z\nu_2 = k$$

$$\nu^2 + \nu_1^2 + \nu_2^2 = 1 \qquad (1.4)$$

For convenience of notation we omit the prime in Eqs. (1.3) and (1.4). The dot indicates differentiation with respect to the dimensionless time $\tau$.

Let us consider the ranges of the parameters involved in the problem. From (1.2) we see that $d$, $k$, $h$, $e$ vary over the infinite range $(-\infty, \infty)$. Expressing the components of the gyration tensor $b_1$, $a_*$ in terms of the inertia tensor and making use of the inequalities relating the moments of inertia, we obtain a restriction on $c$. Setting $A_{11} > A_{22} > A_{33}$, we obtain

$$c = \pm 2 \sqrt{(A_{22} - A_{33}) / A_{11}},$$

and since $A_{22} - A_{33} \leq A_{11}$, the range of the parameter $c$ turns out to be the segment $[-2, 2]$.

**2.** Multiplying the first equation of (1.3) by $y$ and the second by $z$ and then adding them together, we obtain

$$\tfrac{1}{2} (y^2 + z^2)^{\cdot} = \nu_1 z - \nu_2 y \tag{2.1}$$

From (1.4) we obtain

$$\nu_1^2 + \nu_2^2 = 1 - \left(\frac{y^2 + z^2}{c} - h\right)^2, \quad y\nu_1 + z\nu_2 = k - e\left(\frac{y^2 + z^2}{c} - h\right)$$

Substituting these values and (2.1) into the identity, we obtain

$$(y^2 + z^2)(\nu_1^2 + \nu_2^2) - (y\nu_1 + z\nu_2)^2 = (y\nu_2 - z\nu_1)^2$$

and

$$\left(\frac{y^2 + z^2}{2}\right) = -\left((y^2 + z^2)\left[1 - \left(\frac{y^2 + z^2}{c} - h\right)^2\right] - \left[k + eh - e\frac{y^2 + z^2}{c}\right]^2\right)^{1/2}$$

Multiplying the second equation of (1.3) by $y$ and subtracting from it the first equation multiplied by $z$, we arrive at the equation

$$yz^{\cdot} - zy^{\cdot} = -(y^2 + z^2)(y + d) + y\nu_1 + z\nu_2$$

or

$$yz^{\cdot} - zy^{\cdot} = -(y^2 + z^2)\left(y + d + \frac{e}{c}\right) + k + eh$$

Let us introduce the polar coordinates $y = \rho\cos\phi$, $z = \rho\sin\phi$. We then obtain the following system of differential equations for determining $\rho$ and $\phi$:

$$\rho\rho^{\cdot} = -\sqrt{\rho^2[1 - (\rho^2/c - h)^2] - [k + eh - \rho^2 e/c]^2} \tag{2.2}$$

$$\rho^2\varphi^{\cdot} = -\rho^2(\rho\cos\varphi + d + e/c) + k + eh$$

The dependence of $\rho$ on $\phi$ is defined by Eq.

$$\frac{d\varphi}{d\rho} = \frac{\rho^2(\rho\cos\varphi + d + e/c) - k - eh}{\rho\sqrt{\rho^2[1 - (\rho^2/c - h)^2] - [k + eh - (\rho^2 e/c)]^2}} \tag{2.3}$$

The substitution $y = \operatorname{tg}(\phi/2)$ transforms Eq. (2.3) into the Riccati equation

$$\frac{dy}{d\rho} = -\frac{[k + eh - \rho^2(d + e/c) + \rho^3]y^2 + k + eh - \rho^2(d + e/c) - \rho^3}{2\rho\sqrt{\rho^2[1 - (\rho^2/c - h)^2] - [k + eh - \rho^2 e/c]^2}}$$

Setting

$$y = -\frac{du}{d\rho}\,\frac{2\rho\sqrt{\rho^2[1 - (\rho^2/c - h)^2] - [k + eh - \rho^2 e/c]^2}}{u[k + eh - \rho^2(d + e/c) + \rho^3]}$$

in the above equation, we obtain a linear second-order differential equation whose coefficients are polynomials in $\rho$,

$$P_{11}(\rho)\frac{d^2u}{d\rho^2} + P_{10}(\rho)\frac{du}{d\rho} + P_9(\rho)u = 0$$

$$P_9(\rho) = [k + eh - \rho^2(d + e/c) + \rho^3]^2[k + eh - \rho^2(d + e/c) - \rho^3]$$

$${}_{10}(\rho) = -4\rho/c^{-2}\{\rho^9 - 2(d + e/c)\rho^8 + [(2hc - e^2)(d + e/c) + 4(k + eh)]\rho^6 +$$
$$+ c^2[1 - h^2 + 2(e/c)(k + eh)]\rho^5 - 3(2hc - e^2)(k + eh)\rho^4 - 2c^2(k + eh)^2\rho^3 +$$
$$+ c^2(k + eh)[(k + eh)(a - 3e/c) - 2(1 - h^2)] + c^2(k + eh)^3\}$$

$$P_{11}(\rho) = 4\rho^2 [k + eh - \rho^2 (d + e / c) + \rho^3] \{\rho^2 [1 - (\rho^2 / c - h)^2] - [k + eh - \rho^2 e / c]^2\}$$

Setting

$$d + e / c = 0, \; k + eh = 0 \tag{2.4}$$

in Eq. (2.3), we render the latter integrable. We have

$$\rho = \sqrt{A \sin \vartheta + B}, \qquad A = \sqrt{c^2 + {}^1/_4 e^4 - ce^2 h}, \qquad B = {}^1/_2 (2hc - e^2) \tag{2.5}$$

$$\vartheta = \frac{2}{c} [\ln | \operatorname{tg} ({}^1/_2 \varphi + {}^1/_4 \pi) | + n] \tag{2.6}$$

Here $n$ is an integration constant. We note that as $\phi$ varies in the range $(-\pi/2, \pi/2)$, $\vartheta$ varies in the range $(-\infty, \infty)$.

**3.** In order to obtain the geometric picture of motion we must know both the mobile and the stationary hodographs. The angular velocity of the body in the chosen mobile coordinate system can be expressed in terms of the gyration tensor in the following way:

$$\omega_1 = an + b_1 y, \; \omega_2 = b_1 n + a_* y, \; \omega_3 = a_* z$$

Let us convert to dimensionless variables,

$$y', \; z', \; \omega_i' \; (\omega_i = a_* \sqrt{\Gamma / b_1} \omega_i')$$

and set $m' = \sqrt{(b_1/\Gamma)} \, (an - b_1 \lambda_1)/a_*$. This yields

$$\omega_1 = m + {}^1/_2 cy, \;\; \omega_2 = y, \;\; \omega_3 = z \tag{3.1}$$

We have omitted the prime for convenience of notation. Under conditions (2.5) system (3.1) becomes

$$\omega_1 = {}^1/_2 (e + cy), \;\; \omega_2 = y, \;\; \omega_3 = z \tag{3.2}$$

From (3.2) we see that the mobile hodograph lies in the plane $\omega_1 = \frac{1}{2}(e + c\omega_2)$; the projection of the mobile hodograph on the plane $\omega_1 = 0$ is curve (2.5). Let us investigate this curve.

From the condition that $\rho$ is real we infer the need for distinguishing the following cases:

$$\begin{aligned} &1^\circ. \; hc > 0, \; h^2 > 1, \; e^2 < 2c\,(h - \sqrt{h^2 - 1}) \\ &2.^\circ \; hc \geqslant 0, \; h^2 \leqslant 1, \; e^2 \leqslant 2hc \\ &3^\circ. \; hc \geqslant 0, \;\; h^2 \leqslant 1, \;\; e^2 \geqslant 2hc \\ &4^\circ. \; hc \leqslant 0, \; h^2 \leqslant 1 \end{aligned} \tag{3.3}$$

Case $1^\circ$ will be considered specially in Section 6. Let us investigate the remaining cases of (3.3).

1) We begin by considering the projection of the mobile hodograph in the case where the coordinate $\rho$ does not vanish for any value of the polar angle $\phi$. The parameters $c$, $e$, $h$ are here subject to the restrictions of Case $1^\circ$ of (3.3).

The projection of the mobile hodograph is here symmetric with respect to the third coordinate axis, since $\rho(\phi) = \rho(\pi - \phi)$. From (2.5) we find that

$$\sqrt{A - B} \leqslant \rho \leqslant \sqrt{A + B}$$

i.e. that the curve lies in a ring with its center at the origin.

The maximum value of $\rho$ is attained when $\sin \vartheta = 1$; the minimum value corresponds to $\sin \vartheta = -1$. The values of the polar angle $\phi$ are here given by

$$\varphi_l^* = \pm 2 \operatorname{arc} \; \operatorname{tg} \exp (-n + \pi c\,(l + {}^1/_4)) - {}^1/_2 \pi, \;\; l = 0, \; \pm 1, \; \pm 2, \ldots \tag{3.4}$$

$$\varphi_{*l} = \pm \; 2 \operatorname{arc\,tg} \; \exp (-n + \pi c\,(l - {}^1/_4)) - {}^1/_2 \pi \tag{3.5}$$

respectively.

If

$$\varphi_l^0 = \pm \; 2 \operatorname{arc\,tg} \exp (-n + {}^1/_2 \pi c l) - {}^1/_2 \pi \tag{3.6}$$

we have

$$\sin \vartheta = 0, \qquad \rho = \sqrt{B}$$

The maximum points, minimum points, and points (3.6) of the right side of the curve (i.e.

with the plus sign in front of arc tg in Expressions (3.4) - (3.6)) are ordered as follows:

$$\cdots < \varphi^0_{-2} < \varphi_1^* < \varphi^0_{-1} < \varphi_{*_0} < \varphi_0^0 < \varphi_0^* < \varphi_1^0 < \varphi_0^* < \varphi_{*_1} < \varphi^0_2 < \cdots$$

The distances between these points diminish as $\phi$ approaches $\pm\pi/2$, becoming infinitely small in the neighborhood $\pm\pi/2$. This means that there is an infinity of these points in the neighborhood $\pm\pi/2$. The position of points (3.4)-(3.6) depends on the values of the parameters $c$ and $n$. The sign of $\phi_0^0$ coincides with the sign of $n$; $|\phi_0|$ increases with increasing $|n|$, but remains smaller than $\pi/2$. The parameter $c$ characterizes the rate with which points (3.4)-(3.6) tend to $\pm\pi/2$; the smaller the $e$, the slower the approach of these points to $\pm\pi/2$.

Let us take $\phi_0^*$ as the initial point and trace the course of curve (2.5). The point now lies on the outer ring boundary. The coordinate $\rho$ diminishes as $\phi$ increases, reaching the value $\sqrt{B}$ for $\phi = \phi_1^0$; when $\phi = \phi_{*_1}$ the curve touches the inner circle, after which $\rho$ begins to increase; for $\phi = \phi_1{}^*$ curve (2.5) again touches the outer circle. The cycle is then repeated. The behavior of the curve is similar with decreasing $\phi$ (Fig. 1$a$).

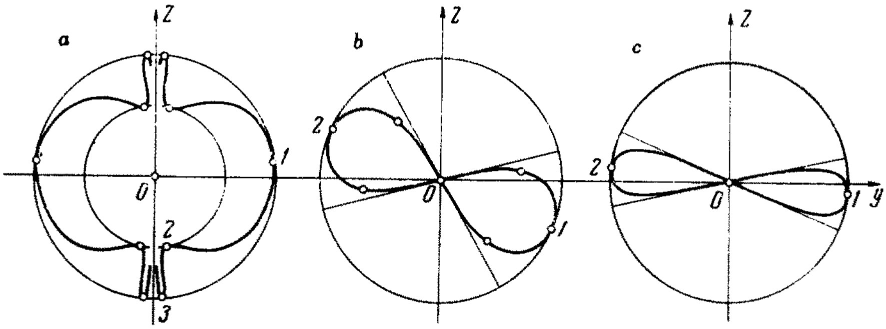


Fig. 1

2) In Cases 2°, 3°, 4° of (3.3) the coordinate $\rho$ vanishes for some $\phi$, and the projection of the mobile hodograph lies inside the disk $\rho \leq \sqrt{A+B}$ with its center at the origin.

When $\vartheta = \vartheta_0 = \arcsin(-B/A)$, curve (2.5) passes through a zero point which is singular for the given curve. In order to investigate the behavior of the curve in the neighborhood of this point, let us consider the time dependence of (2.5) as given by the formulas

$$\rho^{\cdot} = -\cos\vartheta, \qquad \varphi^{\cdot} = -\rho\cos\varphi \tag{3.7}$$

The polar coordinates of a point in this case have the following significance: the coordinate $\rho$ can be either positive or negative, and the polar angle $\phi$ varies from $-\pi/2$ to $\pi/2$.

Let us take $\phi_0^*$ as the initial value of the polar angle and trace the course of curve (2.5) as the time $\tau$ increases from 0. The coordinate $\rho$ has a maximum at the initial instant; the curve touches the circle $\rho = \sqrt{A+B}$. Here $\phi^{\cdot} = -\sqrt{A+B}\cos\phi_0^*$, $\rho^{\cdot} = 0$, and at the next instant $\rho$ begins to diminish with decreasing $\phi$. For

$$\varphi_{0,-1} = 2\,\mathrm{arc\ tg}\exp\left(\tfrac{1}{2}c\vartheta_0 - n\right) - \tfrac{1}{2}\pi \tag{3.8}$$

where $\rho = 0$, the curve passes through the origin, touching the ray $\phi = \phi_{0,-1}$. At this point $\rho^{\cdot} < 0$, which means that $\rho$ becomes negative, and, as we see from (3.7), $\phi$ begins to increase. When $\phi = \phi_0^*$, $\rho^{\cdot} = 0$, the function $\rho$ reaches its minimum value $-\sqrt{A+B}$ (the curve touches the circle $\rho = \sqrt{A+B}$ at this point), and $\rho$ increases with further increases in time. For

$$\varphi_{0,1} = 2\,\mathrm{arc\ tg}\exp\left[\tfrac{1}{2}c(\pi - \vartheta_0) - n\right] - \tfrac{1}{2}\pi \tag{3.9}$$

curve (2.5) again passes through the zero point, touching the ray $\phi = \phi_{0,1}$. The coordinate $\rho$ then becomes positive and $\phi$ begins to decrease. Curve (2.5) finally reaches the initial point when

$$\tau = T = \int_{\vartheta_0}^{\pi-\vartheta_0} \frac{d\vartheta}{\sqrt{A\sin\vartheta + B}}$$

The projection of the mobile hodograph is a closed curve symmetric to the origin, where it has a double point. If the parameters have been chosen in such a way that $\phi_0^* = 0$, curve (2.5) is symmetric with respect to the coordinate axes $y$ and $z$.

The position of the curve varies depending on $n$: the larger the $|n|$, the larger the inclination of the curve relative to the $y$-axis and the smaller the central angle to which the curve is confined. The sign of the angle of inclination is the same as the sign of $n$.

Case 2° of curve (2.5) (Fig. 1*b*) differs from the same curve in Cases 3° and 4° (Fig. 1*c*) in that it assumes the values $\phi_0{}^0$ and $\phi_1{}^0$. It does not assume these values in Cases 3° and 4°, since here $\phi_0{}^0 < \phi_{0,-1}$ and $\phi_1^0 > \phi_{0,1}$.

**4.** We can construct the stationary hodograph by means of Kharlamov's kinematic equations [4],

$$\omega_\zeta(\sigma) = \omega_1(\sigma)\nu_1(\sigma) + \omega_2(\sigma)\nu_2(\sigma) + \omega_3(\sigma)\nu_3(\sigma)$$
$$\omega_\rho{}^2(\sigma) = \omega_1{}^2(\sigma) + \omega_2{}^2(\sigma) + \omega_3{}^2(\sigma) - \omega_\zeta{}^2(\sigma)$$

$$\omega_\rho{}^2(\sigma)\frac{d\alpha}{d\sigma} = \begin{vmatrix} \nu_1(\sigma) & \nu_2(\sigma) & \nu_3(\sigma) \\ \omega_1(\sigma) & \omega_2(\sigma) & \omega_3(\sigma) \\ d\omega_1/d\sigma & d\omega_2/d\sigma & d\omega_3/d\sigma \end{vmatrix}$$

Taking $\phi$ as the independent variable $\sigma$ in these equations, we obtain

$$\omega_\zeta = \frac{1}{2}\left(\frac{\rho^2}{c} - h\right)(c\rho\cos\varphi - e) - eh \tag{4.1}$$

$$\omega_\rho{}^2 = \rho^2 + \frac{1}{4}(e + c\rho\cos\varphi)^2 - \omega_\zeta{}^2 \tag{4.2}$$

$$\frac{d\alpha}{d\vartheta} = \frac{1}{2\omega_\rho{}^2}\left[\frac{eA^2\cos^2\vartheta}{2c\rho} + \rho^2\cos\varphi\left(A\sin\vartheta - \frac{1}{2}c^2\nu_1\right)\right] \tag{4.3}$$

$$\nu_1 = \frac{1}{c}(-e\rho\cos\varphi \pm A\sin\varphi\cos\vartheta) \tag{4.4}$$

Here $\rho$ is defined by Eq. (2.5). The dependence of $\phi$ on $\vartheta$ can be found from (2.6). The sign in (4.4) must be determined from the initial conditions.

Eqs. (4.1) and (4.2) define the meridian of the surface of revolution on which the stationary hodograph lies. It is evident from Eqs. (4.1)-(4.4) that concomitant replacement of $c$, $e$, $h$ by $-c$, $-e$, $-h$ does not alter the shape of the meridian and of curve (2.5), and that the changes in the angle $\alpha$ then proceed in the opposite direction. If we set $-e$ instead of $e$ in (4.1)-(4.4) the shape of the stationary hodograph does not change provided we also set $-\cos\phi$ instead of $\cos\phi$. This means that the portion of the stationary hodograph which corresponds to the right half of curve (2.5) ($\cos\phi > 0$) corresponds to the left half of the curve when we replace $e$ by $-e$.

This implies that need only be considered in cases (3.3) for $c \geq 0$, $e > 0$.

1) Let us consider Case 1° for $c > 0$, $h > 1$, $e^2 < 2c(h - \sqrt{h^2 - 1})$. The meridian line is defined by Eqs. (4.1) and (4.2). The shape of the curve is shown in Fig. 1*a*. As $|\phi| \to \pi/2$ the meridian tends to the circular arc

$$\omega_\rho{}^2 + (\omega_\zeta + c/e)^2 = e^{-2}(c^2 + {}^1/_4 e^4 - hce^2) \tag{4.5}$$

For $\omega_\zeta$ we obtain the expression $\omega_\zeta = -\frac{1}{2}e(\rho^2/c + h)$, whence we see that circle (4.5) lies entirely in the lower half-plane ($\omega_\zeta < 0$).

From (4.3) we find that $d\alpha/d\vartheta$ is positive everywhere except in the ranges $\beta_{*l} < \vartheta < \beta_l{}^*$, where $d\alpha/d\vartheta < 0$ and $\beta_{*l} < \frac{1}{2}\pi(1 + 4\pi l) < \beta_l{}^*$. The values of $\beta_{*l}$, $\beta_l{}^*$ can be obtained from Eq.

$$eA^2\cos^2\vartheta + c\rho^3(2A\sin\vartheta - c^2\nu_1) = 0 \tag{4.6}$$

We note that the lengths of the intervals $(\beta_{*l}, \beta_l^*)$ tends to zero as $\vartheta$ increases to $\infty$.

For sufficiently large $\vartheta$ (we neglect terms of the order sch $(c\vartheta/2 - n)$) Formula (4.3) becomes

$$\frac{d\alpha}{d\vartheta} \sim \frac{e(c^2 + {}^1/_4 e^4 - hce^2)}{4c\rho\omega_\rho^2}\cos^2\vartheta, \qquad \omega_\rho^2 = \rho^2 + \frac{e^4}{4} - \frac{e^2}{4c^2}(\rho^2 + hc)^2$$

From this we find that $\vartheta$ tends to $\infty$ as $\vartheta \to \infty$. The increase in the angle $\alpha$ as $\vartheta$ varies from $2\pi l$ to $2\pi(l+1)$ tends to the constant value $\alpha_0$,

$$\alpha_0 = \frac{e(c^2 + {}^1/_4 e^4 - hce^2)}{4c}\int_0^{2\pi}\frac{\cos^2\vartheta\, d\vartheta}{\rho\omega_\rho^2}$$

as $\vartheta \to \infty$.

Fig. 2*a* shows the meridian for $c = 1.5$, $h = 2$, $e = 0.5$, $n = 1$ and the circular arc which the meridian curve approaches as $\vartheta \to \infty$. We see from the figure that the meridian line very quickly approaches the limiting circle; for these values of the parameters the points of the meridian are already less than 0.01 away from the circle for $\vartheta > 4\pi$.

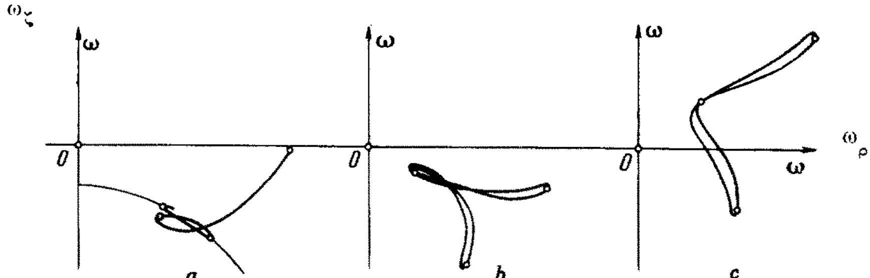


Fig. 2

The stationary hodograph appears in Fig. 3*a*.

2) Let us consider the stationary hodograph when $\rho$ can vanish (Cases $2^\circ$, $3^\circ$, $4^\circ$ of (3.3)).

The meridian is a closed curve whose self-intersection point corresponds to the double point of the mobile hodograph. In Case $2^\circ$ of (3.3) the meridian line has self-intersection points distinct from the latter. Fig. 2*b* shows the meridian of the stationary hodograph for $c = 0.8$, $h = 0.5$, $e = 0.6$, $n = 1$. In Cases $3^\circ$ and $4^\circ$ of (3.3) this singular point is the sole self-intersection point of the meridina curve. The shape of the meridian for $c = 1.8$, $h = -0.6$, $e = 1$, $n = 1$ appears in Fig. 2*c*.

In this case $d\alpha/d\vartheta < 0$ in the range $\beta_{*0} < \vartheta < \beta_0^*$; $\beta_{*0}$, $\beta_0^*$ are the smallest roots of Eq. (4.6). We must bear in mind here that the coordinate $\rho$ can be negative. Let $\vartheta = \beta_{*0}$, $\beta_0^*$ for $\tau = \tau_{*l}$, $\tau_l^*$. Then $\tau_{*l} < \tau_l^0 < \tau_l^*$, where $\tau_l^0$ are the instants at which $\rho$ reaches its minimum value $-\sqrt{A+B}$. Thus, as the time $\tau$ increases, $\alpha$ increases except in the ranges $(\tau_{*l}, \tau_l^*)$ in which the angle $\alpha$ decreases. In the period $T$ in which the variable point of the mobile hodograph traverses the entire hodograph and returns to the initial point, the angle $\alpha$ acquires the increment $\alpha_1$. If $\alpha_1 \neq 0$, then $|\alpha|$ increases without limit as $\tau \to \infty$.

The stationary hodographs for the first (Fig. 3*b*) and second (Fig. 3*c*) cases considered in this Section were constructed assuming that $\phi = \phi_0^*$, $\alpha = 0$ for $\tau = 0$. Figs. 3*b* and 3*c* show one portion of the stationary hodograph corresponding to the change in time from 0 to $T$. The next part of the stationary hodograph results from the preceding part if we rotate the latter by the angle $\alpha_1$.

**5.** We can obtain the picture of motion by rolling the mobile axoid over the stationary one. This requires knowledge of the dependences of $\phi$ and $\vartheta$ on the time $\tau$,

$$\tau = -\int_{\varphi_0^*}^{\varphi}\frac{d\varphi}{\cos\varphi\sqrt{A\sin\vartheta + B}}, \qquad \text{or} \qquad \tau = -\frac{c}{2}\int_{{}^1/_2\pi}^{\vartheta}\frac{d\vartheta}{\sqrt{A\sin\vartheta + B}}$$

We see from this that $\phi$ and $\vartheta$ decrease with time. The stationary hodograph of the problem has already been investigated. The mobile hodograph results when we shift curve (2.5) by the amount $\lambda_1$ in the positive direction along the $y$-axis and then project it onto the plane $\omega_1 = (e + c\omega_2)/2$ parallel to the $x$-axis.

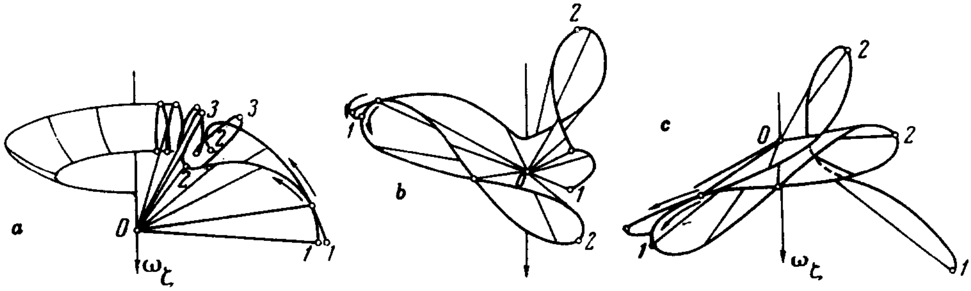


Fig. 3

We choose the initial instant in such a way that $\phi = \phi_0^*$ and the minus sign applies in Expression (4.4). The positions of the axoids at some instant for the cases considered are shown in Fig. 3. The arrows indicate the direction in which the point of contact of the mobile and stationary hodographs. The corresponding hodograph points must coincide during motion. The body executes precessional motion about the vertical axis: if $\alpha_0$ and $\alpha_1$ are commensurate with $2\pi$, then the motion of the body is periodic in Cases $2^\circ$, $3^\circ$, $4^\circ$ of (3.3); in Case $1^\circ$ the motion of the body tends to periodicity as $\tau \to \infty$.

**6.** Let us consider the case $hc > 0$, $h^2 > 1$, $e^2 = 2c(h - \sqrt{h^2 - 1})$. Here $A = 0$ and $\rho = \mathrm{const} = \sqrt{B}$, i.e. curve (2.5) is a circle of radius $\sqrt{B}$. The angle $\phi$ varies according to the law

$$\varphi = \pm 2 \operatorname{arc\,tg} \exp(-\tau \sqrt{B} + n) - {}^1\!/_2\,\pi$$

where the sign and the constant $n$ are chosen from the initial conditions. It is clear that $\phi \to -\pi/2$ as $\tau \to \infty$.

The equations of the stationary hodograph are

$$\omega_\zeta = {}^1\!/_2\,(B/c - h)(c\sqrt{B}\cos\varphi - e) - eh$$

$$\omega_\rho^2 = B + {}^1\!/_4\,(e + c\sqrt{B}\cos\varphi)^2 - \omega_\zeta^2 \tag{6.1}$$

$$\alpha = \frac{8e\sqrt{B}}{4c^2 - e^4}\ln\left|\operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right| + m$$

The value of $m$ must be determined from the initial conditions.

The meridian of the surface of revolution on which the stationary hodograph lies is the straight line

$$\omega_\rho = \left(\frac{4c^2 - e^4}{e^4}\right)^{1/2}\left(\omega_\zeta + \frac{4ecB}{4c^2 - e^4}\right)$$

From (6.1) we find that $\omega_\rho \to 0$ and $\alpha \to -\infty$ as $\phi \to -\pi/2$. For this reason the stationary hodograph is a curve of finite length which winds on the vertical axis an infinite number of times.

Fig. 4 shows the position of the axoids at some instant. The arrows indicate the direction of subsequent motion. Moreover, $\phi = 0$ and $\alpha = 0$ at the initial instant $\tau = 0$, i.e. $m = n = 0$, and the plus sign is taken in front of the expression for $\phi$.

As $\tau \to \infty$ the body tends to uniform rotation with the angular velocity $\omega = 4ecB/(4c^2 - e^4)$ about the $z$-axis which is oriented vertically in stationary space.

In conclusion we note the following.

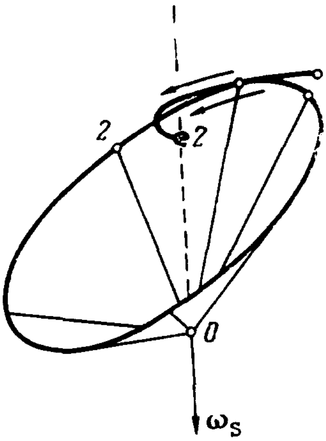

Fig. 4

1. If we set $\lambda = \lambda_1 = \lambda_2 = 0$, then (1.1) yield the equations and integrals in the Hess case. The case where Eq. (2.3) are integrable under conditions (2.4) becomes the familiar case of integrability pointed out by Hess under the condition that the constant sum of the areas is zero.

2. System (2.2) is also integrable in elementary functions in the case where Eq

$$\rho^2\left[1-\left(\frac{\rho^2}{c}-h\right)^2\right]-\left[k+e\iota-\frac{e}{c}\rho^2\right]^2=0$$

has two equal positive roots.

The author is grateful to P.V. Kharlamov for his guidance in the present study.

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Translated by A.Y.